COMBINATIONS OF PYTHAGOREAN TRIANGLES AS GIVING EXERCISES IN COMPUTATION.

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It has been the author's habit to practice his pupils in computation, in order to test and increase their numerical skill, by teaching them to form and calculate triangles whose sides are expressed in whole numbers and whose areas are also so expressed. The simplest way to obtain such triangles is to combine Pythagorean triangles after the example of Hero of Alexandria, eminent as an engineer as well as a mathematician. Hero was the inventor of the Dioptra, an instrument containing the germ of the theodolite. He was also the inventor of the Aeolopile, a precursor of the steam engine, and in all probability also the inventor of the well known formula for the area of a triangle whose sides are given.

Among his works is found the remarkable triangle whose sides are 13, 14, and 15, and whose area is 84. Hero put together this triangle by combining the two Pythagorean triangles whose sides are 5, 12, and 13 and 9, 12, and 15 respectively. The area of the combined triangle 84 is the sum of 30 and 54, the areas of the two Pythagoreans. A combination of the same two Pythagoreans can be made in another way, giving the triangle whose sides are 4, 13, and 15 and whose area is 24, the difference of the areas 30 and 54. These methods can be employed with any pair of Pythagorean triangles which have a common leg; in the case of Hero's pair 12 is the common leg.

We may apply Hero's method to the two Pythagorean triangles whose sides are 10, 8, and 6 and 17, 15, and 8 respectively. We thus get the triangles whose sides are 10, 17, and 21 and 9, 10, and 17 respectively. The areas by Hero's formula are 84 and 36, the sum and difference of the two areas of the right angled triangles 60 and 24 respectively.

This series of processes can be extended to as many pairs of Pythagorean triangles as we please, provided always that the common leg required be brought about by multiplying the parts of one triangle by a whole number.

In Table I, I give a series of Pythagorean triangles, computed by Sir George B. Airy, the late astronomer royal at Greenwich, (*Nature*, 33, 532.) This table seems to me to give a sufficient selection of Pythagorean triangles for practical purposes. It can be extended by the formulae

$$x = 2fab$$
$$y = f(a^2 - b^2)$$
$$z = f(a^2 + b^2)$$

in which a and b are two numbers relatively prime, and f is any whole number or, if both a and b are odd, is the half of any whole number. Thus if $a=7$, $b=1$, $f=\frac{3}{2}$ we shall have

$$x = 21$$
$$y = 72$$
$$z = 75$$

In Table II, I give a selection of triangles whose areas are whole numbers derived from the combination of those given in Table I.

They can be readily tested, and to indicate the method I give the calculation of Triangle No. 14, whose sides are 41, 51, and 58.

$a = 41$	$(s-a) = 34$	$\tan \frac{1}{2} A = \frac{2}{5}$	$\frac{1}{2} A = 21° \; 48' \; 5''$
$b = 51$	$(s-b) = 24$	$\tan \frac{1}{2} B = \frac{17}{30}$	$\frac{1}{2} B = 29 \;\; 32 \;\; 19$
$c = 58$	$(s-c) = 17$	$\tan \frac{1}{2} C = \frac{4}{5}$	$\frac{1}{2} C = 38 \;\; 29 \;\; 35$
$s = 75$		Sum of half angles,	$89° \; 59' \; 59''$

$$\text{Hence } r = \sqrt{\frac{34.24.17}{75}} = \frac{68}{5}$$

In computing the angles I used a five figure table of natural tangents, which is usually sufficient to give the sum of the half angles 90° within a second; tenths of seconds can be obtained with six figure logarithms as a usual thing.

After the tangents of the half angles have been computed as whole numbers or vulgar fractions, I need not say that they can readily be checked by the ordinary trigonometric, or more properly goniometric, formulae, which will show that the sums of the half angles are 90° in each case. Thus in our example

$$\tan \tfrac{1}{2}(A + C) = \tfrac{6}{5} \div \tfrac{17}{25} = \tfrac{30}{17}$$

The writer regards it as an orderly method of teaching Trigonometry to deal with the functions without logarithms before the pupils are required to learn the trigonometric artificialities. In the method which the writer prefers Trigonometry becomes the first mathematical subject of Freshman year, and the extension of Algebra is deferred till Trigonometry is pretty well understood, at least in its elements.

I must defer till another occasion some suggestions relating to the construction of tables of the natural trigonometric functions.

TABLE I.— *Sides of Pythagorean Triangles.*

x	y	z	x	y	z
3	4	5	33	56	65
5	12	13	16	63	65
8	15	17	48	55	73
7	24	25	36	77	85
20	21	29	13	84	85
12	35	37	39	80	89
9	40	41	65	72	97
28	45	53	15	112	113
11	60	61	17	144	145

Table II.— *Sides of Triangles whose Areas also are expressed by Whole Numbers.*

No. of triangle.	a	b	c	No. of triangle.	a	b	c
1	13	14	15	19	52	61	87
2	4	13	15	20	23	61	68
3	10	17	21	21	21	61	68
4	9	10	17	22	53	100	141
5	51	52	53	23	51	53	100
6	4	51	53	24	53	339	364
7	29	75	92	25	53	308	339
8	29	52	75	26	143	339	350
9	13	37	40	27	77	145	156
10	13	30	37	28	145	147	194
11	15	41	52	29	113	145	194
12	15	28	41	30	53	117	136
13	43	61	68	31	53	80	117
14	41	51	58	32	73	143	180
15	33	41	58	33	73	84	143
16	25	74	77	34	45	89	116
17	25	63	74	35	89	116	123
18	61	74	87	36	73	148	195
				37	73	85	148

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