A CORRECTION IN SET THEORY

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PREFACE

The logic and assumptions which today comprise the 'foundations of mathematics' often lead to paradox—the name given to a logical but patently absurd conclusion. One may regard paradox with awe, or one may look for underlying errors.

Orthodox mathematical belief today holds that we may take a solid sphere of any fixed size, divide it into a few pieces, and then reassamble those pieces into two solid spheres, *each* of the same fixed size as the first. This theorem, due to S. Banach and A. Tarski in 1924, has been acclaimed as a triumph of modern methods. Logically similar notions weave through the "new mathematics" taught everywhere today.

In the 1924 paper the authors depend explicitly on the work of F. Hausdorff, who was in turn building on Georg Cantor's theory of sets. So either the sphere surgery can be done, and one equals two, or we had better have another look at Cantor's sets. Here I present an analysis, made possible by modern semantics, of a central fallacy in Cantor's theory. The reader will follow my argument without difficulty if he understands that certain endless sequences of fractions have finite limits; e.g., one-half plus one-quarter plus one-eighth, and so on, never totals more than one, no matter how far extended.

This paper expands on the following topics:

Numbers Generally. The Archimedean test for equality or inequality of two quantities enables us to determine, no matter what anyone may claim, whether some given form actually defines a numerical value, or not.

Scalars. The adjective "real" has traditionally been applied to an important technical class of numbers, confusing students and promulgating philosophical haggles. The exact synonym "scalar" number is adopted for its structural implications.

Endless Convergent Summations. General examples of the "one-half plus one-quarter" type of sequence are introduced, along with the compact modern notation in which we can exactly express them. We find that all of the scalar numbers can be so expressed, under the summation symbol. On the other hand, decimals, while practical and convenient, happen to be inadequate to this task.

Given Numbers. To have a "given number", as it is casually put in the literature, some Jones must give it to some Smith. This human condition limits the possibilities in interesting ways.

Permutations. This is the technical word for what we commonly call arrangements, such as the way in which a set of books might be arrayed along a short shelf, ordered say by authors, or by titles, etc. Simple laws define how much total variety of arrangement is possible given, say, twenty books and a shelf which will hold five.

"New Permutations". With the laws just mentioned, we are in a position to refute immediately any claim that someone could produce a new, unexpected, and unpredictable permutation within an already defined system. From this framework of secure knowledge, we are prepared to examine critically Georg Cantor's famous "proof" that the scalar number system is irreparably disordered and disarrayed.

Cantor's Diagonal Argument. This is the cornerstone of Cantor' century of influence on mathematics. He claims always to generate "new numbers", not already in any conceivable list. From our permutational point of view, his manipulations are not impenetrable. We and Cantor agree on the elements (books and shelf) of a system. We then display to Cantor all permutations of the books taken, say, three at a time (a permutation of length 3). In order to find a "new" permutation, Cantor insists on using a greater length—namely, four. Thereupon we display to him all permutations of length four; he retreats to the claim that he should be allowed five. Before modern semantic methods made the present analysis possible, Cantor's ability to produce "new" arrangements, while unseen he increased the length of the arrays to which he had access, appeared almost magical. His claims were accepted—never explained.

Decimal Expansions. The illusion that Cantor has each time come up with a "new number" involves a misreading of the decimal expressions he uses. By a straightforward inspection of decimals, as well as by a general professional consensus, many scalar numbers cannot be expressed exactly in decimal form. He who fails, through a lack of rigor, to remember the limitations of the decimal system, may imagine that he sees in Cantor's truncated decimal forms the "objective real numbers"; he slides into Cantor's subtle mistake.

The literature contains many dire predictions that parts of higher mathematics "would collapse" if any defect in Cantor's theory were ever found. My personal communications with professionals show that many of them share this fear. What are the facts? Nothing solid in mathematics is going to collapse. Certain passages in special arguments may have to be modified. New, possibly fruitful, insights may result.

INTRODUCTION

Georg Cantor, a German mathematician of the late 19th century, fathered the 'theory of transfinite sets'. His remarkable ideas were scathingly attacked by his contemporary, the illustrious Kronecker. Poincare, considered the leading mathematician of the time, was at first intrigued but later became bitterly disillusioned and declared the thing to be a mathematical disease. But Cantor finally won authoritative acclaim from Hilbert and Bertrand Russell, and carried the day. With rare exceptions, critics have been silent to this day.

Direct results of Cantor's theory now appear even in elementary textbooks, and a long deductive chain runs from it to the 'Banach–Tarski' spheres. In a roundabout but hopefully heuristic way, we want to have a good look at the cornerstone of Cantor's system. We are concerned here only with the set-theoretic issues; the quality and importance of some other work done by Cantor is, I believe, beyond dispute.

Numbers generally. Archimedes first gave formal expression to our common intuition as to what constitutes a number or, rather, how we can tell in some case whether we have a number or not. It is often called Archimedes' principle; it is in the strictest modern sense an operational test. Briefly, if C and D are numbers, then just one of the following three cases holds:

a) C is greater than D ($C > D$),
or b) C is equal to D ($C = D$),
or c) is less than D ($C < D$).

If, for whatever reason in a given case, we find that we cannot determine which of these three condition holds, then it follows that C, or D, or both, does not define a number.

It frequently happens, both in practice and in theoretical work, that we cannot make the Archimedean test because, while we have some information about a number, we do not have enough to define it exactly. That is, we may know a *range* within which the number lies, but nothing more. If C is "4 and a little more", the test may fail. Likewise if C is "4 + a remainder". And particularly, if C is given in an incomplete form, such as $4.32\cdots$, then it is in fact impossible to determine whether C is greater than, equal to, or less than 173/40, for example. The meaning of the familiar " $\cdots$ " can become rather subtle, when we get into endless (infinite) series. A small difference between C and D can make a lot of difference in results. If the expression "C — D" appears in a denominator, then our answer may be negative, or positive, or meaningless, depending upon that "little difference".

The exact definition of a number need not be simply in digits. Not only are operator symbols such as + and / regularly used in defining numbers but purely verbal sentences or longer contexts may be used. Anything we can put in mathematical symbols we can also eventually manage to *say.* "A circle's radius over its circumference" defines in simple English exactly one number; by routine operations we can actually compare "half the reciprocal of pi" with any other given value, for equality or specific inequality, as required.

The full scalar number system. All the numbers referred to hereunder will be *scalar numbers.* One scalar number, or *scalar,* defines exactly one point on a lineal scale which is endless both left and right of a zero point and on which a unit length is assigned. A scalar defines one point. Conversely, any fixed point on the line, even if it has been defined only by geometric operations or otherwise, is associated with one unique scalar. Historically, in cases where a geometric or algebraic definition came first, the development of the required numerical scalar became an urgent need. Such needs have always been met and today the scalar number system is considered complete. All modern theories in calculus and analysis assume and depend upon this fact.

The foregoing statements imply that every scalar number is fully defined, single-valued, fixed and constant. I will rigorously adhere to that connotation. When a domain of definitions, or a range of values, is meant, those terms, along with 'function' and 'variable' will be used. One may argue with considerable justification that the multi-level structure of both mathematical and nonmathematical language make some ambiguity inevitable. I will say that it is difficult to be consistently clear but I will try anyhow.

A scalar number is additive in structure. Any two scalars add to form a single scalar. From the theory of algebraic fields: Addition is postulated, multiplication is repeated addition, subtraction is inverse addition and division is inverse multiplication. As would be expected, the latest forms of scalars to evolve are the most general and have an elaborate composition. However, an organized hierarchy of transformations permits us to express even the earliest and simplest scalars precisely in the latest general forms. For the exact representation of scalars generally, two specific numerical frames are available: Continued fractions, and endless series. It is of interest that we can without loss of meaning interchange the adjectives, to say endless fractions and continued series. Also, many people would prefer "infinite" to "endless"—one may consult the dictionary and his own taste.

Endless convergent summations. Although the continued fractions are of profound number-theory interest, bounded summations of endless series of diminishing fractional terms are more familiar and fully adequate for our purpose. In the powerful notation of modern mathematics, the notion of an endless convergent summation condenses to:

(1) S, a scalar, $= \lim \sum_{i}^{n} \frac{a_i}{b_i}$.

The index i ranges over $i = 1, 2, 3, \cdots$, to and including n. And n has no upper bound. This is important. The symbol n and the complex symbol $n \longrightarrow \infty$ are used interchangeably in the literature, which interchangeability I fully accept, in the above context. There is a context which does not permit n and $n \longrightarrow \infty$ to mean the same thing, and that is when the series a_i/b_i is such that the partial sums do *not* converge. In that case the "limit of the summation" has no meaning, and the summation itself has a numerical value only when n is *fixed.* In the case we are discussing, (1), the series is defined as convergent and the summation has an upper bound and a limit even when n exceeds any bound, and n then exceeds a still higher bound, endlessly.

After such a paragraph, we need examples. Every so-called 'transcendental' scalar can be genetically expressed in the form shown in (1), although the detailed structure of the fraction a_i/b_i may be quite complicated. For a relatively uncomplicated example, we have

(2) e (epsilon, base of nat. logarithms) $= \lim \sum_{i}^{n} \frac{1}{(i-1)!}$.

The summation in (2) can be proven never to exceed 14/5, when n increases without limit.

On the other hand, the innocent-looking fraction 1/i, when part of the summation expression

(3) $\sum_{i}^{n \longrightarrow \infty} \frac{1}{i}$, becomes part of a summation that never stops

growing; it exceeds every bound; hence it would be incorrect to precede it with the term "lim" (limit), or to equate it with any numerical or algebraical symbol whatsoever.

When it is understood that the series converges, we can call $\sum_{i}^{\infty} t_i$ a *closed* or *bound* form, because when t_i is given we have, in a finite number of symbols, all the information necessary to expand the series endlessly. But, of course, to get the closed form, we had to use the finite abbreviation "i" to represent the basic, perfectly regular, endless system of integers 1, 2, · · ·. If i did *not* represent a perfectly regular, pre-known system, it would then be *impossible* to expand $\sum_{i}^{\infty} t_i$ to produce a single, definite numerical scalar value.

Given Numbers. Mathematics textbooks often use the phrase "given the number x . ." Let us expand a little on this phrase, the meaning of which is usually taken to be self-evident. To say that someone, say Jones, has *given a number* to someone else, say Smith, means first of all that they both understand and accept a language and a symbol system. It also means that Jones knows techniques by which he can expand that number to any required degree of accuracy, say to n places, and that Smith, independently, can also expand that number to as many places, and that the two expansions will be identical, term by term. It may be easier, in this case or that, to make decimal expansions, but it can always be done also in common rational fractions. Note that Jones can never give Smith the endless expansion itself—only some closed form. Jones and Smith are limited by the conditions of human communication.

Surprisingly, many of the professional mathematicians whom I have consulted over the years resist and refuse to acknowledge the foregoing operational facts. I have been accused of "philosophizing" about the matter. They have insisted that the endless expansion itself can somehow be completed, communicated, and that it alone constitutes the scalar number. Moreover, the requirement for regularity of development of the expansion is not generally recognized. A professor of mathematics at Michigan, with whom I had already discussed these points in person and established some rapport, sent me a well-drafted letter declaring that the successive digits of the endless decimal expansion of a number could be decided by a function on repeated throws of a pair of dice. He was at first non-plussed when I pointed out that his procedure would predictably produce *different* sequences each time it was performed and so that it was, if anything mathematical, a variable rather than a single, definite number. Soon, however, that reasonable man conceded, "By golly, you're right!" Had he written a textbook *before* that exchange, however, his "formula for producing a scalar number" might very well have gone on to classes and, perhaps, to posterity.

Permutation systems. A permutation is an arrangement of some or all elements of a prescribed set. Repetitions of the same element may or may not be permitted. The word 'sphere' (s,p,h,e,r,e) is a six-place permutation of letters from the 26 of the alphabet. One character, e, is repeated. The total possible variety of such six-place permutations is just 26^6, or exactly 308,915,776, a large but clearly finite number. Most of the arrangements do not spell words, of course. The alphabet itself has a fixed, rather small number of characters. Even the total variety of characters and signs available for printing is limited. Obviously, an 'infinite alphabet' would before long contain unrecognizable symbols and has in fact no serious meaning.

Every expression defining a scalar number is a permutation selected from a fixed 'font' of symbols, including digits and operators such as 2, 7, +, /, etc. Each character in the set is a single, uncompounded, discrete character. The word 'permutation' implies, of course, that the characters gain added significance from their position within the arrangement.

Consider now the two digits 0 and 1 of the binary system. It is a universally acknowledged fact that these two characters, plus the point ".", can define every numerical value that is possible in the standard decimal system. This similarity extends in particular to the issue of endless expansions.

We regard the characters 0 and 1 from a permutational point of view, neglecting for the moment their standing as scalar numbers and digits. However, for continuity in the later argument, each permutation is preceded by a point, so we write .0 1, .0 1 0, .1 1, etc. Since the point appears in the same position in every case, it has no bearing on the variety of possible permutations. Let us now see what is the mathematical meaning of "all possible permutations of 0 and 1, to n places, when $n = 1, 2, 3, \cdots$, and so on without end." We could start with $n = 15$, say, and then pick up the smaller values of n in another order, but this would make systematic examination of all the possibilities more awkward. So we start more naturally with $n = 1$.

Let "pm." stand for "permutation". Then a little experimentation quickly reveals the following structure:

All one-place pms.	All two-place pms.	All three-place pms.	All four-place pms.
.0	.0 0	.0 0 0	.0 0 0 0
.1	.0 1	.0 0 1	.0 0 0 1
	.1 0	.0 1 0	and on so,
	.1 1	.0 1 1	through
		.1 0 0	
		.1 0 1	.1 1 1 1
		.1 1 0	
		.1 1 1	

There is a simple law describing the quantitative growth of permutational variety in the above system. The number of elemental characters is 2. Taking j as index for the number of places, the number of pms. in each row is exactly 2^j. *It cannot be more.* Moreover, there is a simple law for the total number of pms. in all rows up to and including the j-th, and that is just $2^{j+1} - 1$. This last sum shows that all the pms. in the system are technically *countable,* i.e., the pms. may be paired off with the integers $i = 1, 2, 3, \cdots$. We note here the elementary fact that the integers *cannot be summed.* That is, $\sum_{i}^{\infty} i$ is a divergent series and has no total. Likewise, $\sum_{j}^{\infty} 2^j$ is divergent, and indeed at a faster rate than $\sum_{\infty}^{i} i$. Nevertheless, the technical property of countability is preserved. We have sketched the structure of any two-character permutational system. With suitable substitutions in the algebraic variables, the same statements hold for the base-ten, decimal system.

"New" permutations. It 'happens', i.e., it is a consequence of our standard, highly abbreviated system for writing integers, that *every* permutation of the set of base-ten digits, without a point, defines an integer. Some pms. are redundant, eg., 03 = 3, but otherwise there are no "spelling rules" needed.

Referring again to the base-two system, suppose now someone comes along and claims that he has a way of constructing a *new* pm., one *not* included in the countable system we have developed above. To common sense, this appears to be impossible. If his new entry has one place, how can it be other than 0 or 1? *Obviously, it cannot.* If his new entry has two places, how can it be other than 0 0, 0 1, 1 0, or 1 1? *Obviously, it cannot.* Common sense is right. If his new entry has n places, how can it be other than one of the 2^n pms. which develop in our countable system? *It cannot.* Only one 'possibility' remains. Does this orderly, fully developed permutational system suddenly fail if n has no upper bound? Why should it?

Still, someone has made the claim, not only that he can produce a new pm. not in *any* list we can devise, but that he has also thereby proved that pm. systems in general are unorderable, that their members cannot be counted off one-for-one with the integers, and that he thus demonstrates that he knows of an infinity *beyond* infinity. If we try to count *his* pms., he says, the integers will become exhausted. "Yes sir," the head of the mathematics department of a Univ. of Illinois section said matter-of-factly to my face, "The integers will become exhausted." Believe it or not, Georg Cantor made these remarkable claims stick with the world's mathematicians of his time, and they stick unto this day. The effects of the Cantorian grip on the professional mind have to be experienced to be believed.

Cantor's diagonal argument. Here is how he proceeds. Cantor invites us to make a list of permutations, and we are to take the position that our list is so arranged as to include all possible pms. Suppose we start:

1st pm.	.0 0 0
2nd pm.	.0 1 1
3rd pm.	.0 1 0
. . .	. . .

Now, Cantor reminds us, one pm. is defined as distinct from another if it differs from that other *in any one specified place.* We agree to this. "So," he says, "because the first digit of your first pm. is 0, the first digit of my new pm. (usually called z) will be not 0 but 1. Therefore, z is not the same as your first pm." "Now," says Cantor, "the 2nd digit of your 2nd pm. is 1, so I make the 2nd digit of z to be 0. And so z is not your 2nd pm. Since the 3rd digit of your 3rd pm. is 0, I make the 3rd digit of z as 1. Therefore z is not your 3rd pm. And so on. I win." Historically and up to this date, *he has won.* The horrendous "alephs" of his endless infinities thunder through the evening skies of academe "with hooves of steel", as the songwriter put it.

You will doubt that anyone could be deceived by the foregoing brief and transparent manipulations. Nor were they. Cantor's historic presentation was in subtly different terms. No discussion of permutational structures preceded his demonstration. He insists that he is dealing directly with the objects of the "real" (scalar) number system, and that therefore their putative un-orderability and un-countability are a matter of the gravest concern for all of mathematics. He declared that his conclusions were forced upon him by compelling logic, against his own will. He made it stick and, if one believes him, one will also come to believe in the Banach-Tarski spheres.

Let us go back to where Cantor says, "And so on." Because, contrary to Cantor's colleagues and even his critics, we have taken pains to analyze and understand in advance the orderly and endless structure of our permutational system, and we do not have to confuse it with "real numbers", we are in a position to object to his calm, "And so on." "You, Cantor, have not seen our 4th pm. yet. It has only three digits; there is not a "fourth" one to work on; and your "new" pm. z = .1 0 1 certainly *does* appear in our list."

Cantor wants to wave aside our statements. "You don't understand," he says. "All such arrays of digits are endless. You have only shown me three places. Every one knows that all real numbers lead to endless expansions . . ." We ask, "Can't we just talk about the pms. first? All numbers are permutations of digits and other symbols. Don't these laws apply?" And at about this point your Cantor(ian) will declare, "Right now I don't care anything about permutations. What I want to do is prove to you the correctness of the diagonal argument, and you are supposed to have written the real numbers all down here and only the endless decimal form will do and closed forms and finite rationals and things like that simply do not matter. They do not matter because they will not help my argument which can only proceed, when you have already written down all the endless decimals and so on." I do not exaggerate.

Now the refusal of the Cantorians to allow us to develop an orderly listing of all scalars by taking one-symbol pms. first, then two-symbol pms., etc., using whatever conventional digital and operator characters are required, is truly ironic. That is because Cantor himself had earlier made mathematical history by proving that all of the first three orders of scalars, namely integers, rationals, and radicals, could indeed be ordered, and counted, and all had the same "type of infinity". It is worth noting that expositors today describe this feat of Cantor's without ever mentioning that he did it by systematically putting all of the *shortest* forms first; then the next longer, and so on. There is, as you may well conclude, no other way of ordering an endless system. Whether subconsciously or not, these writers manage to avoid displaying short *vs* long forms, by sticking to extremely simple cases in ten-base numerals, or by resorting to algebraic symbols which effectively conceal the lengths of the corresponding numerals. At least a part of their interest is clear: They have written many books giving many solemn proofs, confirmations and consequences of Cantor's argument that, alone among all the scalars, the general form employing the summation operator Σ cannot be ordered and counted. Their *desire* to believe is strong, if not profound. The shibboleth, "The n-th digit of z is different from the n-th digit of the n-th number in the list" acts to paralyze the higher centers. Since the simple scalar numeral "3" does not have a "n-th" digit the Cantorians "expand" it with an infinity of zeroes after the point, for no other reason or purpose than to sustain the illusions of the diagonal argument.

Decimal expansions. Let us look at the definition of an endless decimal as "a definite scalar number". First of all, the literature is in fact replete with descriptions of the properties and limitations of decimal expansions which state that some scalar numbers *cannot* be exactly defined by decimals. Yet, in other references, the endless decimals are given *genetic* status as scalar numbers. The endless decimals may be regarded as the actual objects of the real number system, one authority puts it. Now these declarations involve flat contradictions, of course. We could say that mathematicians are only human, but that would *explain* nothing. There is a reason, indeed a *doctrinal* reason, for this anomalous situation.

I want to say here that our modern system of decimal notation is a truly marvelous mathematical structure, thousands of years in development and brought to its present state only during the Renaissance. (Shortly it will be refined a little more, when the clumsy redundancy of "x 10^e" in so-called scientific notation will be replaced by a suitably positioned single e for the power of ten.) However, decimal fractions, wherein the issue of "expansions" arises, are only one of several possible forms for fractions, and of course everyone who is taught arithmetic is taught common fractions, as well. Now it is an elementary but noteworthy fact that, if we actually restrict ourselves to the ten digits and the point, many common fractions can *never* be exactly shown or 'given' in decimal form. Consider $a/b = .2592\cdots$. This cannot be unambiguously solved for two integers a and b. The best we could say is that a/b is equal to or greater than 162/625 and less than 2593/10,000. Of course this might be good enough for many practical purposes but is not significant in a theory of rational numbers. Recall that the Cantorian diagonal argument implies that a difference in the *n-th* digit is meaningful, no matter whether that might be the 10th digit, or the billionth, or when n increases endlessly.

The integers a and b in the above example can be exactly defined by the *addition* to the decimal system of a conventional but little used symbol called the vinculum (or some equivalent). This is a line over a set of decimal digits meaning that the permutation of digits repeats endlessly in the expansion. In other words, the vinculum is a symbol abbreviating the sentence preceding this one. In many cases, writers modify and circumscribe a numerical expression by a couple of paragraphs of special, one-purpose expository text and then appear to believe that all that complex of meaning actually resides in the numerical itself. It reminds one of the comedians' convention where so many jokes were going about that they were referred to as No. 29, No. 172, etc. When one fellow heard "No. 17" he fell into a fit of laughter, since he had never heard that one before! At any rate, the vinculum is an *addition* to the set of ten digits and the point; it permits us to *drop* one digit from our example and show $a/b = \overline{.259}$, wherefrom a/b is exactly 7/27, and nothing else. Without the vinculum, or some symbolic or linguistic substitution for it, the "endless decimal expansion" cannot define 7/27 or others of that type.

When it comes to so-called irrational numbers, like $\sqrt{7}$, the deficiency of the decimal expansion is far more striking. There is no equivalent of the vinculum for irrationals. Nor can any be invented. For rationals, the vinculum delineates a *pattern* which survives the transcription from the common fraction into the decimal fraction form, and from this pattern the closed form a/b can be exactly recovered. When $\sqrt{7}$ is expanded in decimals, no pattern survives, and hence the original expression cannot be unambiguously recovered. This is all perfectly conventional and well-known, you can check it with anyone, and indeed if you were so inclined could prove it to yourself with some effort and practice. The loss of pattern in an irrational is the cost of expanding it *decimally;* in practice of course it is often advantageous because of the ease with which decimal approximations can be compared with one another, combined with each other, etc.

Perhaps it is not obvious what pattern there actually is in an irrational like $\sqrt{7}$. However, there are many ways to *expand* $\sqrt{7}$ in forms in which the pattern becomes clearly visible and is never lost. Continued fractions, mentioned above, are one. Another is by the expansion of $(6 + 1)^{1/2}$ by the binomial theorem. There are indefinitely many different ways of expanding $\sqrt{7}$ by the two systems mentioned, and there are other systems, too.

So the "endless decimal expansions" cannot give us even a simple radical exactly. But the other systems, including the endless convergent summations, *always* provide closed expressions, not only for radicals but for a still higher form of scalar number usually called "transcendental". These include trigonometric functions, for example, so they are eminently 'practical' sorts of things.

I can assure you that every mathematician will concede the technical accuracy of the foregoing. On some points he may have to do some 'figuring', or long-recalling, but he will concede. As I have pointed out, for mathematicians no less than for the rest of us does a contradiction between two propositions preclude their both being carried in the same head.

So why have the "endless decimal expansions" (or binary expansions, for that matter) been raised to such a status as to be equated with scalar numbers themselves? The answer, I believe, is now clear. Because the open admission that closed forms for scalar numbers, which forms can obviously be ordered and counted, are available in other expansion systems but not in the decimal or binary type, would lead rather quickly to the exposure of the number-theory fallacies in the Cantorian diagonal argument and the deductions made from it. Remember the spheres.